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Weak Chebyshev Spaces and Best L_1-Approximation

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In the first part of this paper it is shown that a large class of weak Chebyshev subspaces of $C[a, b]$ can be considered as a class of generalized spline functions. This generalizes a result of Bartelt (*J. Approx. Theory* **14** (1975), 30–37). In the second part L_1 uniqueness for a large subclass of generalized spline functions is proved. In particular, results of Galkin (*Math. Notes* **15** (1974), 3–8), Strauss ("Numerische Methoden der Approximationtheorie," Band 2, 1973), and Carroll Braess (*J. Approx. Theory* **12** (1974), 362–364) who have shown L_1 uniqueness for subspaces of polynomial spline functions and certain continuously composed Chebyshev subspaces are obtained in this way.

0. INTRODUCTION

Let G denote an n dimensional subspace of $C[a, b]$, the Banach space of continuous real-valued functions on the compact interval $[a, b]$ with the uniform norm. Then G is said to be a *Chebyshev subspace* (*weak Chebyshev subspace*) if each nonzero function in G has not more than $n-1$ zeros (changes of sign) on $[a, b]$. We denote the class of all weak Chebyshev subspaces of $C[a, b]$ of dimension n by W_n.

The prototype of weak Chebyshev subspaces are the subspaces of polynomial spline functions with fixed knots (cf. [6, 7]). In the first part of this paper we present a partial converse to this result referring to a paper of Bartelt [1]. We prove in Theorem 3.1 that under some additional assumptions on a weak Chebyshev subspace G there exists a minimal set of knots $a = x_0 < x_1 < \cdots < x_{s-1} < x_s = b$ such that every g in G has either at most $n_i - 1$ zeros on $[x_{i-1}, x_i]$, where $n_i = \dim G - \dim G_{|[x_{i-1}, x_i]}$, or vanishes identically there. Hence it turns out that G_i is a Chebyshev subspace of $C[x_{i-1}, x_i]$ which implies that G can be considered as a generalized spline subspace of $C[a, b]$. We obtain in this way a generalization of the statement of Theorem 3 in [1]. Furthermore, applying the statement of Theorem 4 in [1] we can easily show (Theorem 3.3) that the

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converse to Theorem 3.1 is also true. Hence we have obtained a characterization of those elements of W_n which can be decomposed by finitely many knots into Chebyshev subspaces.

In order to prove Theorem 3.1 we need some properties of weak Chebyshev subspaces of $C[a, b]$. At first we show in Theorem 1.4 that for every subinterval $[c, d]$ of $[a, b]$ the subspace $\tilde{G} = G|_{[c,d]}$ is a weak Chebyshev subspace of $C[c, d]$. Moreover we examine those subspaces in W_n the elements of which do not vanish on any subinterval of $[a, b]$. Suppose these spaces have *Chebyshev rank* (C-*rank*) less than or equal to k ($k \leqslant n - 1$). This means that the dimension of the set of best approximations of f is not greater than k for each f in $C[a, b]$. We show in Theorem 2.5 that C-rank $\leqslant n - 2$ implies the Chebyshev property and have therefore obtained a new characterization of Chebyshev subspaces.

In addition to the uniform norm let us consider the L_1-norm on $[a, b]$. In the second part of this paper we study the question of whether uniqueness of best L_1-approximation holds for those elements of W_n which satisfy the assumptions of Theorem 3.1. It is well known that if G is a Chebyshev subspace, then every function f in $C[a, b]$ has a unique best L_1-approximation from G (see Rice [8, p. 109]). But contrary to best L_1-approximation, where the Chebyshev property is both sufficient and necessary for uniqueness of best L_1-approximations, there are non-Chebyshev subspaces which also guarantee uniqueness of best L_1-approximations of f for every f in $C[a, b]$. Recently, Galkin [4] and Strauss [14] have established that this phenomenon is even given for subspaces of polynomial spline functions, i.e., for every f in $C[a, b]$ there exists a unique s_0 in $S_{m,k}$, where $S_{m,k}$ denotes the subspace of polynomial spline functions of degree m with k fixed knots, such that

$$\int_a^b |f(x) - s_0(x)|\, dx \leqslant \int_a^b |f(x) - s(x)|\, dx \qquad \text{for every } s \text{ in } S_{m,k}.$$

It is easily verified that $S_{m,k}$ satisfies the assumptions of Theorem 3.1. Taking this fact into consideration we define a large subclass $\tilde{V}_n$ of certain weak Chebyshev subspaces for which the assumptions of Theorem 3.1 are valid. We show (Theorems 4.2 and 4.4) that every G in $\tilde{V}_n$ guarantees uniqueness of best L_1-approximations of f from G for every f in $C[a, b]$. $\tilde{V}_n$ seems to us to be the most important subclass of W_n because every spline subspace $S_{m,k}$ and certain continuously composed Chebyshev subspaces are contained in $\tilde{V}_n$. Hence there follow from our statement the results of Galkin [4] and Strauss [14] for $S_{m,k}$ and of Carroll and Braess [2] for certain continuously composed Chebyshev subspaces. To prove Theorem 4.4 we essentially use a condition established by DeVore and Strauss [15] ensuring uniqueness of best L_1-approximations. This condition is not necessary for uniqueness as we show by two examples.

Furthermore, we present an example where best L_1-approximation of continuous functions, by elements of W_n not in $\tilde{V}_n$, is not unique in general. This is even true for weak Chebyshev subspaces which can be decomposed by knots into Chebyshev subspaces according to Theorem 3.1. For generalized spline subspaces, therefore, uniqueness does not hold in general.

Finally we define a subclass $\bar{\bar{V}}_n$ of W_n the elements of which do not satisfy the assumptions of Theorem 3.1. However, using the same kind of arguments as in the proof of Theorem 4.4 it is easily verified that also every G in $\bar{\bar{V}}_n$ guarantees uniqueness of best L_1-approximations.

1. Restricted Weak Chebyshev Subspaces

We distinguish the following zeros of a function f in $C[a, b]$:

Definition 1.1. A zero $x_0 \in (a, b)$ of f is said to be a *zero with a change of sign* if in each neighborhood of x_0 there exist two points $x_1 < x_0 < x_2$ such that $f(x_1) \cdot f(x_2) < 0$. An isolated zero $x_0 \in (a, b)$ of f is said to be a *double zero* if f does not change sign at x_0. Two zeros x_1, x_2 of f are said to be *separated* if there is an x_0, $x_1 < x_0 < x_2$, such that $f(x_0) \neq 0$. Let $Z(f) = \{x \in [a, b]: f(x) = 0\}$ and let $|Z(f)|$ be the number of zeros of f on $[a, b]$.

We first prove that each subspace $\tilde{G}$ of G obtained by restricting G to a subinterval $[c, d]$ of $[a, b]$ is weak Chebyshev provided that G is weak Chebyshev. To do that we need the following characterization of Jones and Karlovitz [6]:

Lemma 1.2. *Let G be an n-dimensional subspace of $C[a, b]$. Then the following conditions are equivalent*:

(i) $G \in W_n$.

(ii) *Given $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$, there exists a $g \in G$, $g \neq 0$, such that*

$$(-1)^{i-1} g(x) \geqslant 0, \qquad x_{i-1} < x < x_i, \qquad i = 1, \ldots, n.$$

(iii) *If $g_1, g_2, \ldots, g_n$ is a basis of G, then $a \leqslant t_1 < t_2 < \cdots < t_n \leqslant b$, $a \leqslant s_1 < s_2 < \cdots < s_n \leqslant b$ imply*

$$\det |g_i(t_j)| \cdot \det |g_i(s_j)| \geqslant 0.$$

Applying this lemma and the definition of weak Chebyshev subspaces, it is easy to show

LEMMA 1.3. *Let G be an n-dimensional subspace of $C[a, b]$. Then the following conditions hold:*

(i) *If $G \in W_n$, then there exists a $g \in G$ with exactly $n-1$ changes of sign on (a, b).*

(ii) *If $G \notin W_n$, then there exists a $g \in G$ with at least n changes of sign on (a, b).*

We are now in a position to state our first result.

THEOREM 1.4. *Let $G \in W_n$. If $a \leqslant c < d \leqslant b$, then the space $\tilde{G} = G|_{[c,d]}$ is a weak Chebyshev subspace of $C[c, d]$ with dimension less than or equal to n.*

Proof. Let m be the dimension of $\tilde{G}$. We only examine the case when $m < n$. Suppose first that $a = c < d < b$. We show that $\tilde{G}$ is weak Chebyshev. Since $m < n$, there exists a basis $\{g_1, g_2, \ldots, g_n\}$ of G such that $\tilde{G} = \operatorname{span}\{h_1, \ldots, h_m\}$, where $h_i = g_i|_{[a,d]}$, $i = 1, \ldots, m$ and $g_i \equiv 0$ on $[a, d]$, $i = m+1, \ldots, n$.

Let $G_{ad} = \{g \in G: g \equiv 0 \text{ on } [a, d]\}$. Then $G_{ad} = \operatorname{span}\{g_{m+1}, \ldots, g_n\}$. Assume now that $\tilde{G}$ is not weak Chebyshev. This implies that there exists a function $\tilde{g} \in \operatorname{span}\{g_1, \ldots, g_m\}$ with at least m changes of sign on (a, d). By Lemma 1.3 there exists a function $\bar{g} \in G_{ad}$ with at least $n-m-1$ changes of sign on (d, b). Since $\bar{g} \in G_{ad}$, it follows that $\bar{g} \equiv 0$ on $[a, d]$. Then, for sufficiently small $k > 0$, either the function $\bar{g} + k\tilde{g}$ or the function $\bar{g} - k\tilde{g}$ has at least m changes of sign on (a, d), $n-m-1$ changes of sign on (d, b) and a further one on a neighborhood of d. Hence we have found a function in G having at least n changes of sign. But this contradicts the hypothesis of G to be a weak Chebyshev subspace of $C[a, b]$.

Assume now that $a < c < d < b$ and let $\tilde{\tilde{G}} = G|_{[a,d]}$ with $\dim \tilde{\tilde{G}} = r$. Then it follows from the first case that $\tilde{\tilde{G}}$ is a weak Chebyshev subspace of $C[a, d]$. Therefore replacing the subspace G by the subspace $\tilde{\tilde{G}}$ we can conclude as in the first case and get the desired statement.

2. WEAK CHEBYSHEV SUBSPACES AND CHEBYSHEV RANK

In this section we are interested in those subspaces G for which the set of best L_α-approximations of f from G has dimension less than or equal to k ($k \leqslant n-1$) for every f in $C[a, b]$. This property has been investigated by Rubinstein [9] and Zuhovickii [16] in subspaces of $C(Q)$, where Q is a compact Hausdorff space.

DEFINITION 2.1. Let Q be a compact Hausdorff space and let G be a subspace of $C(Q)$ with the uniform norm. G is said to be of *Chebyshev rank*

(C-*rank*) less than or equal to k, if, for each $f \in C(Q)$, the set of best approximations $P_G(f)$ of f from G is at most a k-dimensional polyhedron. G is said to be of C-rank k, if G is of C-rank $\leqslant k$ but not of C-rank $\leqslant k-1$.

Rubinstein has obtained the following characterization of subspaces of C-rank $\leqslant k$.

THEOREM 2.2. *Let G be an n-dimensional subspace of $C(Q)$. For each $f \in C(Q)$, $P_G(f)$ is at most a k-dimensional polyhedron if and only if every $k+1$ linearly independent functions $g_1, \ldots, g_{k+1} \in G$ have at most $n-k-1$ common zeros on Q.*

We are particularly interested in subspaces G of C-rank $\leqslant n-1$. It follows directly from Theorem 2.2 that: G has C-rank $\leqslant n-1$ if and only if for any $x \in Q$ there exists a $g \in G$ with $g(x) \neq 0$.

Considering this property we call a point $x \in Q$ *nonvanishing* with respect to G if there is a $g \in G$ with $g(x) \neq 0$. In the following "with respect to G" will be omitted.

Consequently G has C-rank $\leqslant n-1$ if and only if each $x \in Q$ is nonvanishing. Equivalently, G has C-rank n if and only if there exists an $\tilde{x} \in Q$ with $g(\tilde{x}) = 0$ for every $g \in G$.

Using the statement of Theorem 2.2 and our above conclusions we are able to determine the C-rank of certain subspaces of $C[a, b]$.

EXAMPLE i. Let G be an n-dimensional subspace of $C[a, b]$ and let G contain a positive function. Then G has C-rank $\leqslant n-1$.

EXAMPLE ii. Let $a < x_1 < x_2 < \cdots < x_k < b$ be any partition of $[a, b]$. Then by $S_{m,k} = \operatorname{span}\{1, x, \ldots, x^m, (x-x_1)_+^m, \ldots, (x-x_k)_+^m\}$ we denote the subspace of polynomial spline functions of degree m with k fixed knots $x_1, x_2, \ldots, x_k$. Obviously, $\dim S_{m,k} = m+k+1$. Using the statement of Theorem 2.2 it is easily verified that $S_{m,k}$ has C-rank k.

We are now interested in those $G \in W_n$ which have, for some $k \geqslant 0$, C-rank $\leqslant k$. Furthermore, suppose that G does not contain any function vanishing identically on subintervals of $[a, b]$. We can then show that these subspaces are even Chebyshev provided that $k \leqslant n-2$. To prove this statement we need some properties on weak Chebyshev subspaces of $C[a, b]$.

LEMMA 2.3 (Stockenberg [13]). *Let $G \in W_n$. Then the following conditions hold*:

(i) *If there is a $g \in G$ with n separated, nonvanishing zeros $x_1 < x_2 < \cdots < x_n$, then $g(x) = 0$ for all x with $x \leqslant x_1$ and $x \geqslant x_n$.*

(ii) *No $g \in G$ has more than n separated, nonvanishing zeros.*

LEMMA 2.4 (Sommer and Strauss [11], Stockenberg [12]). *Let $G \in W_n$. Then there exists an $(n-1)$-dimensional weak Chebyshev subspace of G.*

We are now in a position to state our main theorem of this section.

THEOREM 2.5. *Let $G \in W_n$ and let G have* C-rank $\leqslant n-2$. *If no nonzero $g \in G$ vanishes identically on a nondegenerate subinterval of $[a, b]$, then G is a* Chebyshev subspace.

Proof. Since G has C-rank $\leqslant n-2$, it follows from our above considerations that each $x \in [a, b]$ is nonvanishing. Let g be any function in G with at least n zeros. Then by assumption on G all zeros of g are separated, because otherwise g would vanish identically on some subinterval of $[a, b]$. Now applying Lemma 2.3 we can conclude that g has precisely n zeros $x_1 < \cdots < x_n$ and $g(x) = 0$ for all x with $x \leqslant x_1$ and $x \geqslant x_n$. Then our assumption on G implies that $x_1 = a$ and $x_n = b$. Thus we have shown that G is a Chebyshev subspace on $[a, b)$ and also on $(a, b]$, i.e., each $g \in G$ has at most $n-1$ zeros on these intervals.

Suppose now that G is not Chebyshev on $[a, b]$. Then arguing as above we obtain a function $\bar{g} \in G$ such that $\bar{g}$ has precisely n zeros $a = x_1 < x_2 < \cdots < x_n = b$, where at $x_2, \ldots, x_{n-1}$ there are changes of sign of $\bar{g}$. By Lemma 2.4, there is a basis $\{g_0, g_1, \ldots, g_{n-1}\}$ of G such that $g_0, g_1, \ldots, g_i$ span an $(i+1)$-dimensional weak Chebyshev subspace of G, $i = 0, \ldots, n-1$. Without loss of generality let g_i have exactly i changes of sign on (a, b), $i = 0, \ldots, n-1$. We distinguish two cases.

Case I (n even, say, $n = 2p$). Then for sufficiently small $c > 0$, the functions $\bar{g} - cg_{2k}$, $k = 0, \ldots, p-1$, have n zeros on $[a, b]$. Now arguing as above we can conclude that $g_{2k}(a) = g_{2k}(b) = 0$, $k = 0, \ldots, p-1$. Similarly, for sufficiently small $c > 0$, the functions $g_{2k} \pm cg_{2k-1}$, $k = 1, \ldots, p-1$, have at least $2k$ changes of sign on (a, b) and, in case $g_{2k-1}(a) \neq 0$ or $g_{2k-1}(b) \neq 0$, at least one further change of sign at a neighborhood of a or b, respectively. However, these functions belong to the weak Chebyshev subspaces span$\{g_0, g_1, \ldots, g_{2k}\}$ and, therefore, each such function can have at most $2k$ changes of sign. This implies that $g_{2k-1}(a) = g_{2k-1}(b) = 0$, $k = 1, \ldots, p-1$. Thus there exist $n-1$ linearly independent functions $g_0, \ldots, g_{n-2}$ in G vanishing at a and b. But this contradicts the hypothesis that G has C-rank $\leqslant n-2$.

Case II (n odd). We proceed analogously.

This completes the proof of Theorem 2.5.

Remark i. It follows immediately from the first part of the proof of Theorem 2.5 that C-rank $n-1$ already implies that G is a Chebyshev subspace on $[a, b)$ and also on $(a, b]$.

Remark ii. The statement of Theorem 2.5 corrects the statement of Theorem 2 of Bartelt [1]. His theorem was stated for subspaces containing the constants (recall that such subspaces have C-rank $\leqslant n-1$), however, in [11] a counterexample has been presented.

3. Generalized Spline Subspaces

Now we are able to prove that weak Chebyshev subspaces, under appropriate hypotheses, can be considered as generalized spline subspaces.

Theorem 3.1. *Let $G \in W_n$ and let each $x \in [a, b]$ be nonvanishing. Assume also that there exists a $\delta > 0$, such that, if $g \in G$ and $g \equiv 0$ on $[c, d] \subset [a, b]$, then $d - c \geqslant \delta$. Then there exist knots $a = x_0 < x_1 < \cdots < x_s = b$ such that the subspaces $G_i = G|_{[x_{i-1}, x_i]}$ are Chebyshev subspaces of $C[x_{i-1}, x_i]$ with dimension n_i, $i = 1, \ldots, s$.*

Proof. First we follow the lines of Bartelt [1, Theorem 3]: There exist points $a = y_0 < y_1 < \cdots < y_r = b$ such that no nonzero $g \in \tilde{G}_i$ vanishes identically on a subinterval of $[y_{i-1}, y_i]$, where $\tilde{G}_i = G|_{[y_{i-1}, y_i]}$. Using Theorem 1.4 we have that every $\tilde{G}_i$ is a weak Chebyshev subspace with dimension m_i. We also show that $\tilde{G}_i$ has C-rank $\leqslant m_i - 1$. Suppose that, for some $i \in \{1, \ldots, r\}$, this property is not given. Then there exists an $\tilde{x} \in [y_{i-1}, y_i]$ such that $g(\tilde{x}) = 0$ for every $g \in \tilde{G}_i$. This implies that $g(\tilde{x}) = 0$ for every $g \in G$, a contradiction.

Thus we have shown that, for $i = 1, \ldots, r$, $\tilde{G}_i$ is an m_i-dimensional weak Chebyshev subspace of $C[y_{i-1}, y_i]$ with C-rank $\leqslant m_i - 1$ and no nonzero $g \in \tilde{G}_i$ vanishes identically on a subinterval of $[y_{i-1}, y_i]$. Therefore it follows from the remark following the proof of Theorem 2.5, for $i = 1, \ldots, r$, $\tilde{G}_i$ is a Chebyshev subspace on $(y_{i-1}, y_i]$ and also on $[y_{i-1}, y_i)$. If $\tilde{G}_i$ is even Chebyshev on $[y_{i-1}, y_i]$, then we are done. But if, for some $i \in \{1, \ldots, r\}$, $\tilde{G}_i$ is not Chebyshev there, then we divide $[y_{i-1}, y_i]$ and choose a further knot $\tilde{y}_i = (y_{i-1} + y_i)/2$. This implies that both $G|_{[y_{i-1}, \tilde{y}_i]}$ and $G|_{[\tilde{y}_i, y_i]}$ are Chebyshev subspaces with dimension m_i. This we may do for all intervals $[y_{i-1}, y_i]$, $i = 1, \ldots, r$, on which $\tilde{G}_i$ is not Chebyshev. We end up with a set of knots $a = x_0 < x_1 < \cdots < x_s = b$ such that, for $i = 1, \ldots, s$, $G_i = G|_{[x_{i-1}, x_i]}$ is Chebyshev with dimension n_i. This completes the proof of Theorem 3.1.

Remark i. Following the lines of Theorem 3 in [1] it is easily verified that the set of knots constructed in the proof of Theorem 3.1 is a minimal set.

Remark ii. If we assume that G has C-rank $\leqslant n-2$, then the intervals $[y_{i-1}, y_i]$ must not be divided. Instead we apply Theorem 2.5. To do this we

show that all subspaces $\tilde{G}_i$ have C-rank $\leqslant m_i - 2$, $i = 1,\ldots, r$. For a fixed i let $g_{m_i+1}, g_{m_i+2},\ldots, g_n$ be linearly independent functions in G such that $g_j \equiv 0$ on $|y_{i-1}, y_i|$, $j = m_i + 1,\ldots, n$. Now suppose that $\tilde{G}_i$ fails to have C-rank $\leqslant m_i - 2$. This implies that there exist $m_i - 1$ functions $g_1, g_2,\ldots, g_{m_i-1} \in G$ which are linearly independent on $|y_{i-1}, y_i|$ and have two common zeros there. Therefore the functions $g_1,\ldots, g_{m_i-1}, g_{m_i+1},\ldots, g_n$ have at least two common zeros on $|y_{i-1}, y_i|$. However, this contradicts the hypothesis that G has C-rank $\leqslant n - 2$. Thus we have shown that $\tilde{G}_i$ has C-rank $\leqslant m_i - 2$, $i = 1,\ldots, r$. Now using Theorem 2.5 we conclude that $\tilde{G}_i$ is a Chebyshev subspace of $C|y_{i-1}, y_i|$ with dimension m_i, $i = 1,\ldots, r$.

Remark iii. The essential difference between Theorem 3 in [1] and Theorem 3.1 is as follows: In Theorem 3 in [1] it is assumed that the constants belong to G. In Theorem 3.1 we only need that each $x \in |a, b|$ be nonvanishing. This is really a weaker assumption as the following example shows: Let $G = \operatorname{span}\{g_0, g_1\} \subset C|0, 3|$, where the functions g_0, g_1 are defined by $g_0(x) = x^2 - 3x$ and

$$\begin{aligned} g_1(x) &= x - 1, && \text{if} \quad x \in |0, 1|, \\ &= 0, && \text{if} \quad x \in |1, 2|, \\ &= x - 2, && \text{if} \quad x \in |2, 3|. \end{aligned}$$

Then it is easily verified that G is weak Chebyshev and satisfies the assumptions of Theorem 3.1. But G does not contain any positive function.

Remark iv. The condition on the length of the zero intervals in Theorem 3.1 cannot be omitted as has been shown in [1].

We now present an example showing that because of the choice of the knots $y_1, y_2,\ldots, y_{r-1}$ it is generally necessary to divide some intervals and to add further knots. Let $G = \operatorname{span}\{g_1, g_2, g_3\} \subset C|-1, 3|$, where the functions g_1, g_2, g_3 are defined by $g_1(x) = 1$,

$$\begin{aligned} g_2(x) &= x(1 - x^2), && \text{if} \quad x \in |-1, 1|, \\ &= 0, && \text{if} \quad x \in |1, 3|, \end{aligned}$$

and

$$\begin{aligned} g_3(x) &= x^2, && \text{if} \quad x \in |-1, 1|, \\ &= 1, && \text{if} \quad x \in |1, 3|. \end{aligned}$$

Then it is easily shown that G is weak Chebyshev and has C-rank 2. Now following the lines of the proof of Theorem 3 in [1] we get $y_1 = 1$ as the only knot. However, $\tilde{G}_1 = G|_{|-1,1|}$ is not Chebyshev on $|-1, 1|$. Therefore we

have to divide the interval $[-1, 1]$ a further time and obtain, finally, the knots $x_1 = 0$ and $x_2 = 1$.

It turns out that a similar result as in Theorem 3.1 can be obtained for a certain subclass of those elements of W_n which have C-rank n.

THEOREM 3.2. *Let $G \in W_n$. Let $\bar{x} \in [a, b]$ and let $g(\bar{x}) = 0$ for every $g \in G$. Assume also that there exists a $\delta > 0$ such that, if $g \in G$ and $g \equiv 0$ on $[c, d] \subset [a, b]$, then $d - c \geqslant \delta$. Then there exist knots $a = x_0 < x_1 < \cdots < x_s = b$ such that the subspaces $G_i = G|_{[x_{i-1}, x_i]}$ are weak Chebyshev with dimension n_i $(n_i \geqslant 0)$, $i = 1, \ldots, s$. Furthermore, no nonzero $g \in G_i$ vanishes identically on a nondegenerate subinterval of $[x_{i-1}, x_i]$.*

Arguing as in the proof of Theorem 3.1 we can easily verify the above statements. Weak Chebyshev subspaces of the same type as those in Theorem 3.2 have been investigated by Sommer and Strauss [11] and Stockenberg [13].

Now we show that the converse of Theorem 3.1 is also true.

THEOREM 3.3. *Let $a = x_0 < x_1 < \cdots < x_s = b$ be knots on $[a, b]$. For $i = 1, \ldots, s$, let G_i be a Chebyshev subspace of $C[x_{i-1}, x_i]$ with dimension n_i $(n_i \geqslant 1)$. Let $G = \{g \in C[a, b]: g|_{[x_{i-1}, x_i]} \in G_i, i = 1, \ldots, s\}$. Then G is a weak Chebyshev subspace of $C[a, b]$ with dimension $n = \sum_{i=1}^{s} n_i - (s - 1)$. Furthermore, each $x \in [a, b]$ is nonvanishing.*

Proof. Bartelt [1, Theorem 4] has proved that G is weak Chebyshev under the additional assumption that the constants are contained in every G_i. Since each Chebyshev subspace contains a positive function (see Karlin and Studden [7, p. 28]), the first sentence of the conclusion of Theorem 3.3 follows directly from Theorem 4 in [1]. Because of the existence of a positive function in G, the second sentence follows immediately, too.

We call weak Chebyshev subspaces such as the one constructed in Theorem 3.3 "continuously composed Chebyshev subspaces" (CC subspaces). In particular, if $n_i = m$, $i = 1, \ldots, s$, then G is a subspace of spline functions with $s - 1$ fixed knots of multiplicity $m - 1$.

4. BEST L_1-APPROXIMATION

In addition to the uniform norm let the L_1-norm be endowed on $C[a, b]$. For every subspace G of $C[a, b]$ consider the *set of best L_1 approximations* of a function f from G

$$P_G^1(f) = \{g_0 \in G: \|f - g_0\|_1 = \inf\{\|f - g\|_1: g \in G\}\}.$$

We need the following condition established by DeVore for one-sided L_1-approximation and by Strauss [15] for L_1-approximation.

DEFINITION 4.1. Let G be a subspace of $C[a, b]$. Let each $g \in G$ have only finitely many separated zeros. We say that G satisfies *condition* A if, for every nonzero function $g_0 \in G$ and every finite subset $Z = \{z_i\}_{i=1}^{r}$ of $Z(g_0) \cap (a, b)$ $(r \in \mathbb{N})$, there exists a nonzero function $g_1 \in G$ satisfying:

(i) $(-1)^i g_1(x) \geqslant 0$, $x \in [z_{i-1}, z_i]$, $i = 1, \ldots, r+1$, $z_0 = a$, $z_{r+1} = b$, and

(ii) if $g_0 \equiv 0$ on an open subset M of $[a, b]$, then $g_1 \equiv 0$ on M. (Recall that $Z(g_0)$ is the set of zeros of g_0 on $[a, b]$.)

If $r = 1$, then $Z = \varnothing$. Therefore, the existence of a nonnegative function in G is required.

Strauss [15] has proved the following result:

THEOREM 4.2. *Let G be an n-dimensional subspace of $C[a, b]$ satisfying condition* A. *Then $P_G^1(f)$ is a singleton for every $f \in C[a, b]$.*

Now we show that for a large class of weak Chebyshev subspaces condition A is satisfied. By Theorem 4.2, therefore, uniqueness of best L_1-approximations follows. We also show that, in particular, the subspaces of polynomial spline functions and the CC subspaces belong to that class. We define

$$V_n = \{G \in W_n: G \text{ fulfills the hypotheses of Theorem 3.1}\}.$$

Recall that, by Theorem 3.1, V_n contains exactly those elements $G \in W_n$ which can be decomposed by finitely many knots into Chebyshev subspaces. Now let $G \in V_n$ and let $a = x_0 < x_1 < \cdots < x_s = b$ be knots of G as in Theorem 3.1. We define for any $i, j \in \{0, 1, \ldots, s\}$, $i < j$

$$G_{ij} = \{g \in G: g \equiv 0 \text{ on } [x_i, x_j]\}, \qquad \dim G_{ij} = m_{ij}.$$

In general, the subspace G_{ij} is not weak Chebyshev. However, we are able to define a subclass of V_n for which every G_{ij} has this property.

$$\tilde{V}_n = \{G \in V_n: |\operatorname{bd} Z(g)| \leqslant m_{ij} \text{ for every } g \in G_{ij},\ i, j \in \{0, \ldots, s\},\ i < j\}.$$

Here $\operatorname{bd} Z(g)$ denotes the set of all boundary points of $Z(g)$ and $|\operatorname{bd} Z(g)|$ denotes the number of all boundary points of $Z(g)$.

We are now in position to present some important examples of elements of $\tilde{V}_n$.

EXAMPLE 1. Let $m, k \in \mathbb{N}$ with $m + k + 1 = n$. Let $a = x_0 < x_1 < \cdots < x_{k+1} = b$ be a partition of $[a, b]$. Consider the subspace $S_{m,k}$ of polynomial

spline functions of degree m with k fixed simple knots $x_1, \ldots, x_k$ as defined in Section 2. It is well known (cf., e.g., Karlin and Studden [7, p. 18]) that $G = S_{m,k} \in W_n$. Now it is easily verified that for any $i, j \in \{0, \ldots, k+1\}$, $i < j$, $G_{ij} = \operatorname{span}\{(x_1 - x)_+^m, (x_2 - x)_+^m, \ldots, (x_i - x)_+^m, (x - x_j)_+^m, (x - x_{j+1})_+^m, \ldots, (x - x_k)_+^m\}$. This implies that $\dim G_{ij} = m_{ij} = k + i - j + 1$. By counting zeros of splines as in Schumaker [10, p. 288–289], it immediately follows that $|\operatorname{bd} Z(g)| \leqslant k + i - j + 1 = m_{ij}$, for every $g \in G_{ij}$. Hence $S_{m,k} \in \tilde{V}_n$. This result is also true for knots with multiplicity less than $m + 1$.

EXAMPLE ii. Let G be constructed as in Theorem 3.3. Then it is easy to show that, for any $i, j \in \{0, \ldots, s\}$, $i < j$, G_{ij} has a basis consisting of: For $p = j+1, \ldots, s$, $n_p - 1$ functions vanishing identically on $[a, x_{p-1}]$ and linearly independent on $[x_{p-1}, x_p]$ and, for $p = 1, \ldots, i$, $n_p - 1$ functions vanishing identically on $[x_p, b]$ and linearly independent on $[x_{p-1}, x_p]$. Hence $\dim G_{ij} = \sum_{p=j+1}^{s} n_p + \sum_{p=1}^{i} n_p - (s - j + i) = m_{ij}$.

Let $g \in G_{ij}$. Then $|\operatorname{bd} Z(g)| \leqslant n_p - 1$ on $[x_{p-1}, x_p]$, $p = 1, \ldots, i, j+1, \ldots, s$. This implies that $|\operatorname{bd} Z(g)| \leqslant m_{ij}$ which shows that G belongs to $\tilde{V}_n$.

L_1-Uniqueness for subspaces of polynomial spline functions has been recently shown by Galkin [4] and Strauss [14] and for some special CC subspaces by Carroll and Braess [2]. We are now able to establish L_1-uniqueness for each subspace G in $\tilde{V}_n$. Then, in particular, from our theorem there follow all of these results.

We need the following fundamental lemma:

LEMMA 4.3. *Let $G \in \tilde{V}_n$. Then, for any $i, j \in \{0, \ldots, s\}$, $i < j$:*

(i) *G_{ij} is weak Chebyshev with dimension m_{ij};*

(ii) *For every function $g_1 \in G_{ij}$ there is a function $\tilde{g}_1 \in G$ such that $\tilde{g}_1 = g_1$ on $[x_j, b]$ and $\tilde{g}_1 \equiv 0$ on $[a, x_j]$;*

(iii) *For every function $g_2 \in G_{ij}$ there is a function $\tilde{g}_2 \in G$ such that $\tilde{g}_2 = g_2$ on $[a, x_i]$ and $\tilde{g}_2 \equiv 0$ on $[x_i, b]$.*

Proof. (i) Suppose that G_{ij} is not weak Chebyshev for some (i, j). Then there exists a $g_0 \in G_{ij}$ with at least m_{ij} changes of sign on (a, b). Since $g_0 \equiv 0$ on $[x_i, x_j]$, $|\operatorname{bd} Z(g_0)| \geqslant m_{ij} + 1$, in contradiction to the hypothesis $G \in \tilde{V}_n$. We prove now (ii); (iii) follows analogously. Let $R_{ij} = \{g|_{[x_j, b]}; g \in G_{ij}\}$, $\dim R_{ij} = r_{ij}$. Obviously, $r_{ij} \leqslant m_{ij}$. The statement will be proved if we can show that $m_{0j} = r_{ij}$, where $m_{0j} = \dim G_{0j}$. Using Theorem 1.4 we conclude that R_{ij} is weak Chebyshev. Therefore by Lemma 1.3 there exists a function $g_1 \in G_{ij}$ with $r_{ij} - 1$ changes of sign on (x_j, b). This implies that $|\operatorname{bd} Z(g_1)| \geqslant r_{ij}$ on $[x_j, b]$. If $r_{ij} = m_{ij}$, then it follows from the hypothesis on G_{ij} that $|\operatorname{bd} Z(g_1)| \leqslant m_{ij} = r_{ij}$ and, therefore, $g_1 \equiv 0$ on $[a, x_j]$. This implies that $g_1 \in G_{0j}$. Then it follows from the hypothesis on G_{0j} that $|\operatorname{bd} Z(g_1)| \leqslant$

m_{0j}. Thus we have shown that $m_{ij} = r_{ij} = |\text{bd } Z(g_1)| \leqslant m_{0j}$. Hence observing that $m_{0j} \leqslant m_{ij}$ we get the desired equality $m_{0j} = r_{ij}$.

Now assume that $r_{ij} < m_{ij}$. Then there exist exactly $m_{ij} - r_{ij}$ linearly independent functions in G_{ij} vanishing identically on $]x_j, b[$. This implies that $\dim G_{is} = m_{is} = m_{ij} - r_{ij}$. Let G_{is} be spanned by the functions h_1, $h_2, \ldots, h_{m_{is}}$. If there exists a $\tilde{g} \in G_{is}$ such that $\tilde{g} = g_1$ on $]a, x_i[$, then $\tilde{g} - g_1$ is an element of G_{0j} satisfying $|\text{bd } Z(\tilde{g} - g_1)| \geqslant r_{ij}$. Since $G_{0j} \subset G_{ij}$, it follows that $m_{0j} \leqslant r_{ij}$. Hence the preceding arguments show that $m_{0j} = r_{ij}$. If there does not exist any $\tilde{g} \in G_{is}$ such that $\tilde{g} = g_1$ on $]a, x_i[$, then the space

$$\tilde{G} = \text{span}\{h_1, h_2, \ldots, h_{m_{is}}, g_1\}|_{]a, x_i[}$$

has dimension $m_{is} + 1$. By Lemma 1.3 there exists a $g_2 \in G_{ij}$ with at least m_{is} changes of sign on (a, x_i). This implies that $|\text{bd } Z(g_2)| \geqslant m_{is} + 1$ on $]a, x_i[$. As $g_2 = \sum_{i=1}^{m_{is}} a_i h_i + b g_1$, $|\text{bd } Z(g_2)| = |\text{bd } Z(g_1)| \geqslant r_{ij}$ on $]x_i, b[$ and, therefore, $|\text{bd } Z(g_2)| \geqslant m_{is} + 1 + r_{ij} = m_{ij} - r_{ij} + 1 + r_{ij} = m_{ij} + 1$. But this contradicts the hypothesis that $|\text{bd } Z(g)| \leqslant m_{ij}$ for every $g \in G_{ij}$. (Recall that $h_1, \ldots, h_{m_{is}}$ span an m_{is}-dimensional weak Chebyshev subspace and, therefore, the coefficient b in the representation of g_2 must be nonzero.) This completes the proof of Lemma 4.3.

We are now in position to state the main result of this section.

THEOREM 4.4. *Let $G \in \tilde{V}_n$. Then G satisfies condition* A.

Proof. Let $g_0 \in G$ and let $Z \subset Z(g_0) \cap (a, b)$ be a finite set. We distinguish two cases:

Case 1. Let g_0 not vanish identically on any nondegenerate subinterval of $]a, b[$. Let $Z = \{z_p\}_{p=1}^{r}$ $(r \in \mathbb{N})$. Then it follows from Lemma 2.3 that $r \leqslant n$. Using Lemma 2.4 we find an r-dimensional weak Chebyshev subspace of G, and by Lemma 1.2, therefore, there is a nonzero $\tilde{g} \in G$ such that $(-1)^p \tilde{g}(x) \geqslant 0$, $x \in]z_{p-1}, z_p[$, $p = 1, \ldots, r$, $z_0 = a$, $z_r = b$.

Case 2. Let g_0 vanish on a subinterval of $]a, b[$.

We have to distinguish, once more, three cases:

Case i. Let $I =]x_i, x_j[$, $a < x_i < x_j \leqslant b$ such that $g_0 \equiv 0$ on I and g_0 does not vanish identically on any nondegenerate subinterval of $]a, x_i[$. Using Lemma 4.3 we find a $\tilde{g}_0 \in G_{is}$ with $\tilde{g}_0 = g_0$ on $]a, x_j[$. By hypothesis on G_{is}, $|\text{bd } Z(\tilde{g}_0)| \leqslant m_{is}$ and, therefore, $|\text{bd } Z(g_0)| \leqslant m_{is} - 1$ on $]a, x_i[$. Then the assumption on g_0 implies that g_0 has at most $m_{is} - 1$ zeros on $]a, x_i[$. Let $Z \cap (a, x_i) = \{z_p\}_{p=1}^{r}$ $(r \in \mathbb{N})$. Then $r \leqslant m_{is}$. Using Lemmas 2.4 and 1.2 we obtain a nonzero function $\tilde{g} \in G_{is}$ such that

$$(-1)^p \tilde{g}(x) \geqslant 0, \qquad x \in]z_{p-1}, z_p[, \quad p = 1, \ldots, r, \quad z_0 = a, \quad z_r = x_i.$$

In particular, $\tilde{g} \equiv 0$ on $]x_i, b[$.

Case ii. Let $I = [x_i, x_j]$, $a \leqslant x_i < x_j < b$ such that $g_0 \equiv 0$ on I and g_0 does not vanish identically on any nondegenerate subinterval of $[x_j, b]$. Here we can conclude as in Case ii.

Case iii. Let $I_1 = [a, x_h]$ and $I_2 = [x_i, x_j]$, $x_h < x_i$, such that $g_0 \equiv 0$ on $I_1 \cup I_2$ and g_0 does not vanish identically on any nondegenerate subinterval of $[x_h, x_i]$. Using Lemma 4.3 we find a $\tilde{g}_0 \in G_{is}$ with $\tilde{g}_0 = g_0$ on $[a, x_i]$. This implies that the linear subspace $\tilde{G} = G_{0h} \cap G_{is}$ has dimension $d \geqslant 1$.

We show that $\tilde{G}$ is even weak Chebyshev. Suppose that $\tilde{G}$ fails to be weak Chebyshev. Then there exists a $g_1 \in \tilde{G}$ with at least d changes of sign on (x_h, x_i). Since $g_1 \in \tilde{G}$, it follows that $g_1 \equiv 0$ on $[a, x_h] \cup [x_i, b]$. This implies that $|\operatorname{bd} Z(g_1)| \geqslant d + 2$. Now using that $\tilde{G} \subset G_{0h}$ we conclude that $|\operatorname{bd} Z(g_1)| \leqslant m_{0h}$. Therefore $m_{0h} \geqslant d + 2$. Hence there exist $m_{0h} - d$ functions in G_{0h} linearly independent on $[x_i, b]$ and a function $g_2 \in G_{0h}$ with at least $m_{0h} - d - 1$ changes of sign on (x_i, b). Therefore, for sufficiently small $c > 0$, either the function $g_1 + cg_2$ or the function $g_1 - cg_2$ has at least $m_{0h} - d - 1$ changes of sign on (x_i, b), at least d changes of sign on (x_h, x_i) and a further one at a neighborhood of x_i. This implies that one of these functions has at least m_{0h} changes of sign on (x_h, b). But this contradicts the hypothesis that G_{0h} is weak Chebyshev. Thus we have shown that $\tilde{G}$ is a weak Chebyshev subspace.

Next we show that g_0 has at most $d - 1$ zeros on (x_h, x_i). Assume that $|Z(g_0)| \geqslant d$ on (x_h, x_i). Then $g_0(x_h) = g_0(x_i) = 0$ implies that $|Z(g_0)| \geqslant d + 2$ on $[x_h, x_i]$. From $\tilde{g}_0 = g_0$ on $[a, x_i]$ and $\tilde{g}_0 \in G_{is}$ it follows that $|Z(g_0)| \leqslant m_{is}$ on $[x_h, x_i]$. Therefore $d < m_{is} = \dim G_{is}$. Now using the fact that $\tilde{G} \subset G_{is}$ we find $m_{is} - d$ functions in G_{is} linearly independent on $[a, x_h]$ and a function $\bar{g} \in G$ with at least $m_{is} - d - 1$ changes of sign on (a, x_h). Let m be the number of all zeros of g_0 on $[x_h, x_i]$ and r_1 the number of all common zeros of g_0 and $\bar{g}$ on $[x_h, x_i]$. We classify the other $m - r_1$ zeros of g_0 on $[x_h, x_i]$ as follows:

Let r_2 be the number of all double zeros with the property that for each of these zeros there exists a neighborhood U such that $g_0(x)\,\bar{g}(x) \geqslant 0$ for every $x \in U$.

Let r_3 be the number of all double zeros with the property that for each of these zeros there exists a neighborhood U such that $g_0(x)\,\bar{g}(x) \leqslant 0$ for every $x \in U$.

Let r_4 be the number of changes of sign.

In the case when $\bar{g}(x_h) \neq 0$, the zero x_h of g_0 is not considered in the above classification, because by Definition 1.1, x_h is neither a double zero nor a zero with a change of sign of g_0. Thus we have

$$\begin{aligned} m &= r_1 + r_2 + r_3 + r_4 + 1, && \text{if } \bar{g}(x_h) \neq 0, \\ &= r_1 + r_2 + r_3 + r_4, && \text{if } \bar{g}(x_h) = 0. \end{aligned}$$

We distinguish two cases:

Case i ($r_2 > r_3$ or $r_2 < r_3$, respectively). Without loss of generality let $r_2 > r_3$. Then for sufficiently small $c > 0$ the function $g_0 - c\bar{g}$ has at least $m_{is} - d - 1$ separated zeros on (a, x_h) and at least $r_1 + r_4 + 2r_2 \geqslant r_1 + r_2 + r_3 + r_4 + 1 \geqslant m$ separated zeros on $]x_h, x_i[$. This implies that $|\operatorname{bd} Z(g_0 - c\bar{g})| \geqslant m + m_{is} - d - 1 \geqslant d + 2 + m_{is} - d - 1 = m_{is} + 1$ on $]a, x_i[$. As $g_0 = \tilde{g}_0$ on $]a, x_i[$, $|\operatorname{bd} Z(\tilde{g}_0 - c\bar{g})| \geqslant m_{is} + 1$. However, the function $\tilde{g}_0 - c\bar{g}$ is an element of G_{is} and, therefore, we get a contradiction to the hypothesis that $|\operatorname{bd} Z(\tilde{g}_0 - c\bar{g})| \leqslant m_{is}$.

Case ii ($r_2 = r_3$). Without loss of generality let $g_0 \bar{g} \geqslant 0$ on some neighborhood of x_h (otherwise we take $-\bar{g}$). Then for sufficiently small $c > 0$ the function $g_0 - c\bar{g}$ has at least $m_{is} - d - 1$ separated zeros on (a, x_h) and at least

$$r_1 + r_4 + 2r_2 + 1 = r_1 + r_2 + r_3 + r_4 + 1 = m, \qquad \text{if} \quad \bar{g}(x_h) \neq 0,$$

$$r_1 + r_4 + 2r_2 = r_1 + r_2 + r_3 + r_4 = m, \qquad \text{if} \quad \bar{g}(x_h) = 0,$$

separated zeros on $]x_h, x_i[$ (in the case when $\bar{g}(x_h) \neq 0$ there exists at least one zero in some neighborhood of x_h). This implies that $|\operatorname{bd} Z(g_0 - c\bar{g})| \geqslant m_{is} - d - 1 + m \geqslant m_{is} - d - 1 + d + 2 = m_{is} + 1$, a contradiction as has been shown in Case i.

Thus we have proved that $|Z(g_0)| \leqslant d - 1$ on (x_h, x_i). (Note that this property holds for every function $g \in \tilde{G}$). Now let $Z \cap (x_h, x_i) = \{z_p\}_p^{r-1}$ ($r \in \mathbb{N}$). Then, as in Case 1, we get a function $\tilde{g} \in \tilde{G}$ satisfying condition A. Thus we have shown that G satisfies condition A.

In general, condition A is not necessary for uniqueness of best L_1-approximations. We show this by two examples. In particular, we will see that the weak Chebyshev property is not sufficient for L_1-uniqueness. For this we need a characterization of L_1-uniqueness established by Cheney and Wulbert [3].

THEOREM 4.5. *Let G be a subspace of $C[a, b]$. Then the following conditions are equivalent:*

(i) *For any $f \in C[a, b]$ the set $P_G^1(f)$ is at most a singleton.*

(ii) *If for any function $f \in C[a, b]$ with $0 \in P_G^1(f)$, there exists a function $g \in G$ with $Z(g) \supset Z(f)$, then $g \equiv 0$.*

For special subspaces of $C[a, b]$ we can present Corollary 4.6. To do this let G be a subspace and let

$$\Sigma_G = \left\{ s\colon [a, b] \to \mathbb{R}\colon s(x) \in \{0, -1, 1\},\right.$$

$$\left. \int_a^b g(x)\, s(x)\, dx = 0 \text{ for every } g \in G \right\}.$$

By a theorem of Hobby and Rice [5], Σ_G is always a nonempty set.

COROLLARY 4.6. *Let G be a subspace of $C[a, b]$. Let no nonzero $g \in G$ vanish identically on a nondegenerate subinterval of $[a, b]$. Assume also that the Lebesgue measure $\mu(Z(g)) = 0$ for every nonzero $g \in G$. Then the following conditions are equivalent:*

(i) *If there are functions $f \in C[a, b]$ and $g \in G$ such that $0 \in P_G^1(f)$ and $Z(g) \supset Z(f)$, then $g \equiv 0$.*

(ii) *If there are functions $f \in C[a, b]$ and $g \in G$ such that $s = \operatorname{sgn} f \in \Sigma_G$ and $Z(g) \supset Z(s)$, then $g \equiv 0$.*

Proof. Let $0 \in P_G^1(f)$ and $g_0 \in G$, $g_0 \not\equiv 0$, with $Z(g_0) \supset Z(f)$. By the well-known characterization theorem for best L_1-approximation (see Rice [8, p. 103]), we obtain

$$\left| \int_a^b g(x) \operatorname{sgn} f(x)\, dx \right| \leqslant \int_{Z(f)} |g(x)|\, dx \qquad \text{for every} \quad g \in G.$$

Then, since $\mu(Z(g_0)) = 0$ and $Z(g_0) \supset Z(f)$, this is equivalent to

$$\left| \int_a^b g(x) \operatorname{sgn} f(x)\, dx \right| \leqslant \int_{Z(g_0)} |g(x)|\, dx = 0 \qquad \text{for every} \quad g \in G.$$

This implies that $s = \operatorname{sgn} f \in \Sigma_G$ and $Z(g_0) \supset Z(s)$, $g_0 \not\equiv 0$.

Remark. It turns out that this characterization is also true for arbitrary one- and two-dimensional subspaces of $C[a, b]$. But we do not know if it is also valid in the higher dimensional case.

Using the above results we can prove that condition A is not necessary for L_1-uniqueness.

EXAMPLE 1. Let $G = \operatorname{span}\{g_0\} \subset C[-1, 1]$, where g_0 is defined by

$$g_0(x) = -\tfrac{1}{2}x, \qquad \text{if} \quad x \in [-1, 0],$$
$$= x, \qquad \text{if} \quad x \in [0, 1].$$

Then it is easily verified that there exists no function $f \in C[-1, 1]$ for which $s = \operatorname{sgn} f \in \Sigma_G$ and $Z(s) \subset Z(g_0)$. Using Theorem 4.5 and Corollary 4.6 we

can then conclude that every $f \in C|-1, 1|$ has a unique best L_1 approximation from G. However, condition A is not satisfied.

EXAMPLE ii. Let $G = \operatorname{span}\{g_0, g_1\} \subset C|0, 4|$ and let the functions g_0 and g_1 be defined by $g_0(x) = 1$ and

$$\begin{aligned} g_1(x) &= x - \alpha, && \text{if} \quad x \in |0, \alpha|, \\ &= 0, && \text{if} \quad x \in |\alpha, 4 - \alpha|, \\ &= x - (4 - \alpha), && \text{if} \quad x \in |4 - \alpha, 4|, \end{aligned}$$

where $1 < \alpha < 2$. Let some $f \in C|0, 4|$ be given such that $s_0 = \operatorname{sgn} f \in \Sigma_G$. Then s_0 has at least one zero on $|0, \alpha)$ and at least one on $(4 - \alpha, 4|$. However, there does not exist any nonzero $g \in G$ with zeros on $|0, \alpha)$ and on $(4 - \alpha, 4|$. Therefore Corollary 4.6 shows that G guarantees L_1-uniqueness. Now setting $x_0 = 0$, $x_1 = \alpha$, $x_2 = 4 - \alpha$, $x_3 = 4$ we cannot find any nonzero $g \in G$ such that $(-1)^i g(x) \geqslant 0$, $x \in |x_i, x_{i+1}|$, $i = 0, 1, 2$, which implies that condition A is not satisfied.

The next example shall illustrate that, contrary to subspaces of polynomial spline functions, L_1-uniqueness does not generally hold for subspaces of generalized splines.

EXAMPLE. Let $G = \operatorname{span}\{g_0, g_1\} \subset C|0, 4|$, where $g_0(x) = 1$ and g_1 is as in Example ii, with $\alpha = 1$. This shows that $G \in W_2$. Now setting $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$ we can conclude that the subspace $G_i = G|_{[x_{i-1}, x_i]}$, $i = 1, 2, 3$, is Chebyshev. Therefore $G \in V_2$. However, $G \notin \tilde{V}_2$, since $G_{12} = \operatorname{span}\{g_1\}$ and $|\operatorname{bd} Z(g_1)| = 2$. Now it is easily shown that the function s_0 defined by

$$\begin{aligned} s_0(x) &= 1, && \text{if} \quad x \in |0, 1), \\ &= 0, && \text{if} \quad x = 1, \\ &= -1, && \text{if} \quad x \in (1, 3), \\ &= 0, && \text{if} \quad x = 3, \\ &= 1, && \text{if} \quad x \in (3, 4|, \end{aligned}$$

is contained in Σ_G. Obviously, $Z(g_1) \supset Z(s_0)$. Hence the statements of Corollary 4.6 and Theorem 4.5 show that G does not guarantee L_1-uniqueness.

Considering the proofs of Lemma 4.3 and Theorem 4.4 it turns out that all arguments occurring there can also be applied to those weak Chebyshev

subspaces which fulfill the hypotheses of Theorem 3.2. To show this we define the subclass $\bar{V}_n$ of W_n by

$$\bar{V}_n = \{G \in W_n: G \text{ fulfills the hypotheses of Theorem 3.2}\}.$$

For each $g \in \bar{V}_n$, let $a = x_0 < x_1 < \cdots < x_s = b$ be knots as in Theorem 3.2. Let G_{ij} and m_{ij} be defined as in the case when $G \in V_n$. Let

$$\bar{\bar{V}}_n = \{G \in \bar{V}_n: |\operatorname{bd} Z(g)| \leqslant m_{ij} \text{ for every } g \in G_{ij},\ i, j \in \{0, \ldots, s\},\ i < j,$$
$$\text{and } |Z(g)| \leqslant n - 1 \text{ on } [a, b) \text{ and on } (a, b] \text{ for every } g \in G \text{ which}$$
$$\text{does not vanish on a nondegenerate subinterval of } [a, b]\}.$$

Then Theorem 4.7 is an immediate consequence of the arguments occurring in the proofs of Lemma 4.3 and Theorem 4.4.

THEOREM 4.7. *Let $G \in \bar{\bar{V}}_n$. Then G satisfies condition* A.

Therefore by Theorem 4.2, every $G \in \bar{\bar{V}}_n$ guarantees L_1-uniqueness. However, simple examples show that this property is not generally given if $G \in \bar{V}_n$.

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